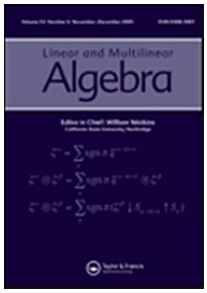

Evolution coalgebras on chicken populations

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Evolution coalgebras on chicken populations

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ABSTRACT
We introduce a coalgebra structure for chicken populations, the evolution coalgebra of a chicken population, and review its main properties. A notion of simplicity is also considered together to a characterization of simple coalgebras of chicken populations given in terms of the strongly connectedness of their attached digraphs, a notion also introduced here.

KEYWORDS
Evolution coalgebra; Chicken population; Digraph

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1. Introduction

The coalgebraic approach to genetic populations was firstly considered by Tian and Li [17], for populations ruled by Mendel's laws, aimed to provide an algebraic model to describe the backwards inheritance of the genetic information between generations. The cubic matrix representation of genetic coalgebras, later considered in [8], contributed to solve some of the question that remained open in [17] setting also a connection between genetic coalgebras and Markov processes [9]. The structure of genetic coalgebras has been recently studied in terms of the strongly connectedness of their attached oriented trigraphs [11].

On the other hand, although genetic populations not obeying Mendel's laws, but based on the self-replication of their individuals, have been till now mainly described in terms of evolution algebras (we refer the reader to Tian's foundation of evolution algebras given in [16]), it has been recently shown that considering such populations from the coalgebraic viewpoint is also possible [13].

Here we attempt to extend that point of view considered in [13] to chicken populations. Chicken populations were described by Labra, Ladra and Rozikov in [4] (see also [6]) as a simplified example of bisexual populations [5] with only one existing male type. Such a reduction, from bisexual to chicken populations, is mainly motivated by the complexity of the cubic structure matrices that appear attached to bisexual populations. These structure matrices become much simpler square matrices under the

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assumption of the population to be of chicken-type.

Keeping in mind that cubic matrices arise naturally attached to genetic coalgebras [8] and considering chicken populations as a preliminary stage towards the coalgebraic representation of bisexual populations, we recover here the definition of evolution coalgebra of a chicken population, already sketched in [13], to deepen into their main algebraic properties. Knowing which the main coalgebraic properties of chicken populations are, is also the first stage to later consider when such coalgebras can be endowed with genetic realization [17] and therefore to study their connection to Markov processes, as done in [9] for those coalgebras arising from Mendelian populations.

This paper is organized as follows. After this introductory section, the second section recalls the notion of evolution algebra of the chicken population (for short, EACP) [4,6] and briefly summarizes the main results known for EACPs. The third section focuses first on general coalgebras [1,15] to recall then some already known coalgebraic structures with genetic significance, as for instance, coalgebras with genetic realization introduced by Tian and Li [17] and evolution coalgebras introduced in [13]. In the fourth section, we recover the notion of evolution coalgebra of a chicken population (for short ECCP) sketched in [13], to study then the algebraic properties of ECCPs. We also settle the relationship between ECCPs and EACPs through their dual structures.

The fifth section considers simple ECCPs. We remark that here simplicity of ECCPs is understood in the sense of not containing proper subcoalgebras that can be identified with smaller chicken populations. To tackle this problem we bring back the notion of trigraph for coalgebras with genetic realization [11] to introduce digraphs for ECCPs. Then the simplicity of ECCPs is established in terms of the strongly connectedness of a distinguished subdigraph.

2. Chicken populations.

Evolution algebras of chicken populations were introduced in [4] (see also [6]) as a particular case of evolution algebras for bisexual populations [5] where only one male type (called roaster) is allowed. In what follows, unless otherwise stated, we will consider K to be a field of characteristic not 2.

Definition 2.1. [4, Definition 2.1] An *evolution algebra of a chicken population* (EACP for short) is a $(n+1)$-dimensional K-algebra $\mathcal{C}$ with (natural) basis $\mathcal{B} = \{h_1, \ldots, h_n, r\}$ and multiplication given by $h_i r = r h_i = \frac{1}{2}(\sum_{j=1}^{n} a_{ij} h_j + b_i r)$, $i = 1, \ldots, n$, and zero otherwise (i.e. $h_i h_j = h_j h_i = 0$ for all $i, j = 1, \ldots, n$ and $r^2 = 0$).

Reduction from bisexual to chicken populations is aimed to obtain a more easy to handle matrix representation of the structure constants attached to the algebra multiplication. Indeed the structure constants of any $(n+1)$-dimensional EACP can be gathered into a $n \times (n+1)$ matrix

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right)$$

EACPs of arbitrary dimension were also considered in [4, Definition 2.1].

As noted in [4], it follows from [5, Theorem 4.1] that, EACPs are commutative, but not associative or even power-associative in general. Moreover EACPs are not unital [4,

Proposition 2.14]. Subalgebras of EACPs were also introduced in [4], following Tian's notion of evolution subalgebras [16], and later studied in [7]. EACPs of dimension 2 and also those 3-dimensional EACPs with $dim_K(\mathcal{C}^2) \neq 1$ were classified in [4, Proposition 6.1, Theorem 6.2]. A complete classification of 3-dimensional complex EACPs was later given in [2].

3. Coalgebras.

We refer the reader to [15] for basic results on coalgebras. See also [13] for those notions more closely related to coalgebras appearing in genetics.

3.1. Coalgebras.

A *coalgebra* C is a K-vector space with a linear map $\Delta : C \to C \otimes C$ called *comultiplication*. A coalgebra (C, Δ) is *coassociative* if $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ and *cocommutative* if $\tau\Delta = \Delta$, where $\tau : C \otimes C \to C \otimes C$ denotes the twist map $\tau(a \otimes b) = b \otimes a$ for all $a, b \in C$. A coalgebra (C, Δ) is *counital* if it is endowed with a linear map (*counit*) $\varepsilon : C \to K$ such that $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$.

3.2. Subcoalgebras and coideals.

Let (C, Δ) be a coalgebra. A subspace D of C is a *subcoalgebra* if $\Delta(D) \subseteq D \otimes D$. A subspace I of C is a *left coideal* (resp. a *right coideal*) if $\Delta(I) \subseteq C \otimes I$ (resp. if $\Delta(I) \subseteq I \otimes C$) and a *coideal* if $\Delta(I) \subseteq I \otimes C + C \otimes I$ (and $\varepsilon(I) = 0$ if C is counital). Given a subset S of a coalgebra C we denote by $subcoalg(S)$ the subcoalgebra of C generated by S, that is, the intersection of all coalgebras of C containing S.

Example 3.1. The dual vector space $A^* = Hom_K(A, K)$ of any finite-dimensional associative K-algebra A with multiplication $m : A \otimes A \to A$, has a coalgebra structure (A^*, Δ) given by $\Delta = \rho^{-1}m^*$, where ρ denotes the isomorphism $A^* \otimes A^* \cong (A \otimes A)^*$. Here ρ is an isomorphism by the finite-dimensionality of A [15]. We remark here that the coalgebra structure on A^* needs of the finite-dimensionality of A. Otherwise the finite dual A° should be considered. However the dual vector space C^* of any coassociative coalgebra C can always be endowed with an associative algebra structure [15, Proposition 1.1.2].

3.3. Baric coalgebras.

A *baric* coalgebra (C, Δ, ϕ) is a (non-necessarily counital) coalgebra (C, Δ) with a *character* ϕ, that is, a nonzero linear map $\phi : C \to K$ such that $(\phi \otimes \phi)\Delta = \phi$. Characters of the coalgebra C are the idempotents of its dual algebra C^* [10, Proposition 1].

3.4. Coalgebras with genetic realization.

The notion of coalgebra with genetic realization was considered in [17] for Mendelian populations. A (finite-dimensional) real coalgebra C has *genetic realization* if it has a (natural) basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ such that the comultiplication constants of the

basis elements are nonnegative and satisfy $\sum_{i,j=1}^{n} \beta_{ij}^{k} = 1$ for all $k = 1, 2, \ldots, n$, where $\Delta(e_k) = \sum_{i,j=1}^{n} \beta_{ij}^{k} e_i \otimes e_j$ for all $k = 1, 2, \ldots, n$. Examples of coalgebras with genetic realization can be found in [10].

Coalgebras with genetic realization are not necessarily coassociative or cocommutative, but can admit characters. Their characters were described in [10]. The study of coalgebras with genetic realization has been based on their connection to cubic matrices with stochastic properties [8].

3.5. Evolution coalgebras.

A new model of coalgebra motivated by self-reproduction processes occurring in non-mendelian populations has been recently introduced in [13]. An *evolution coalgebra* is a K-coalgebra (here K denotes any field) having a (natural) basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ with comultiplication $\Delta(e_k) = \sum_{i=1}^{n} \beta_{ii}^{k} e_i \otimes e_i$ for all $k = 1, 2, \ldots, n$. To endow any (real) evolution coalgebra with genetic realization it then suffices to require the nonnegativity of the β_{ii}^{k}'s and also $\sum_{i=1}^{n} \beta_{ii}^{k} = 1$ for all $i, k = 1, 2, \ldots, n$.

4. Coalgebras on chicken populations.

In this section, following [13], we introduce a coalgebraic framework for chicken populations. All coalgebras considered here are defined over a field K of characteristic not 2.

Definition 4.1. An *evolution coalgebra of a chicken population* (for short ECCP) is a $(n+1)$-dimensional K-coalgebra C with (natural) basis $\mathcal{B} = \{h_1, \ldots, h_n, r\}$ and comultiplication given by:

$$\Delta(h_i) = \sum_{j=1}^{n} \frac{1}{2} \beta_j^i (h_j \otimes r + r \otimes h_j), \quad i = 1, \ldots, n;$$

$$\Delta(r) = \sum_{j=1}^{n} \frac{1}{2} \beta_j (h_j \otimes r + r \otimes h_j).$$

Proposition 4.2. *ECCPs are cocommutative.*

Proof. If follows straightforwardly from Definition 4.1. □

Following [8] the comultiplication constants of any ECCP C can be arranged into a cubic $(n+1) \times (n+1) \times (n+1)$ matrix $\widetilde{P}$, that unfolded by its frontal slices (see [12]) can be written as:

$$\widetilde{P} = \left(\begin{array}{c|c|c|c} \widetilde{P}_{::1} & \cdots & \widetilde{P}_{::n} & \widetilde{P}_{::n+1} \end{array} \right)$$

where (for $i = 1, \ldots, n$)

$$\widetilde{P}_{::i} = \frac{1}{2}\left(\begin{array}{ccc|c} 0 & \cdots & 0 & \beta_1^i \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \beta_n^i \\ \hline \beta_1^i & \cdots & \beta_n^i & 0 \end{array}\right) \quad \text{and} \quad \widetilde{P}_{::n+1} = \frac{1}{2}\left(\begin{array}{ccc|c} 0 & \cdots & 0 & \beta_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \beta_n \\ \hline \beta_1 & \cdots & \beta_n & 0 \end{array}\right).$$

However, here as in [13] in the case of algebras, this cubic matrix can be replaced by a simpler rectangular $n \times (n+1)$ matrix of the form (up to a scalar $\frac{1}{2}$):

$$P = [A|b] = \left(\begin{array}{ccc|c} \beta_1^1 & \cdots & \beta_1^n & \beta_1 \\ \vdots & \ddots & \vdots & \vdots \\ \beta_n^1 & \cdots & \beta_n^n & \beta_n \end{array}\right).$$

Definition 4.3. Given an ECCP C we refer to the $n \times (n+1)$ matrix $P = [A \mid b]$ to be its *associated structure matrix*.

Remark 1. The matrix $P = [A \mid b]$ is close to be (up to a scalar $\frac{1}{2}$) the accompanying matrix (see [10]) of the cubic matrix $\widetilde{P}$. Indeed, being ECCPs cocommutative (see Proposition 4.2), the cubic matrix $\widetilde{P}$ is $(1,2)$-symmetrical, so that their accompanying matrices coincide (a proof similar to that of [10, Lemma 2] works here). Thus we have:

$$\widetilde{P}_{(i)} = \widetilde{P}_{(j)} = \frac{1}{2}\left(\begin{array}{ccc|c} \beta_1^1 & \cdots & \beta_1^n & \beta_1 \\ \vdots & \ddots & \vdots & \vdots \\ \beta_n^1 & \cdots & \beta_n^n & \beta_n \\ \hline \sum_{l=1}^n \beta_l^1 & \cdots & \sum_{l=1}^n \beta_l^n & \sum_{l=1}^n \beta_l \end{array}\right).$$

Then (up to scalar $\frac{1}{2}$) $P = [A \mid b]$ is a submatrix of the accompanying matrices of $\widetilde{P}$.

Proposition 4.4. *ECCPs are not in general coassociative.*

Proof. Consider the $(2+1)$-dimensional ECCP C with natural basis $\{h_1, h_2, r\}$ and comultiplication given by

$$\begin{aligned} \Delta(h_1) &= \frac{1}{2}(h_2 \otimes r + r \otimes h_2) \\ \Delta(h_2) &= \sum_{j=1}^{2} \frac{1}{2}\beta_j^2(h_j \otimes r + r \otimes h_j), \quad \text{for arbitrary } \beta_1^2, \beta_2^2 \in K \\ \Delta(r) &= \frac{1}{2}(h_1 \otimes r + r \otimes h_1). \end{aligned}$$

Now it is a straightforward checking to prove that $(\Delta \otimes id)\Delta(r) \neq (id \otimes \Delta)\Delta(r)$. Hence C, with the given comultiplication, is not coassociative. □

Proposition 4.5. *ECCPs are not counital.*

Proof. Let C be an ECCP and assume $\varepsilon : C \to K$ is a counit for C. Write then $\varepsilon(h_i) = \alpha_i$, $i = 1, \ldots, n$ and $\varepsilon(r) = \rho$ for the elements of a natural basis $\mathcal{B} = \{h_1, \ldots, h_n, r\}$

of C. Then applying the counit condition $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$ to the natural basis elements we get:

$$\beta_i \rho = 0, \qquad i = 1, \ldots, n, \tag{1}$$
$$\beta_1 \alpha_1 + \cdots + \beta_n \alpha_n = 2, \tag{2}$$
$$\beta_j^i \rho = 0, \qquad i \neq j, i, j = 1, \ldots, n, \tag{3}$$
$$\beta_i^i \rho = 2, \qquad i = 1, \ldots, n, \tag{4}$$
$$\beta_1^i \alpha_1 + \cdots + \beta_n^i \alpha_n = 0, \qquad i = 1, \ldots, n. \tag{5}$$

By (4) we have $\rho \neq 0$. But if $\rho \neq 0$, then by (1) $\beta_i = 0$ for all $i = 1, \ldots, n$ which contradicts (2). As a result there is no counit for C. □

Theorem 4.6. *Any nontrivial character ϕ of an ECCP C is of the form $\phi(h_i) = \alpha_i$, $i = 1, \ldots, n$ and $\phi(r) = \rho$, with $\rho \neq 0$ and such that $a^T = (\alpha_1, \ldots, \alpha_n)$ satisfies*

$$a^T b = 1 \tag{6}$$
$$a^T A = \frac{1}{\rho} a^T \tag{7}$$

where $P = [A \mid b]$ is the $n \times (n+1)$ matrix associated to C.

Proof. Let $\phi \in Hom_K(C, K) = C^*$ be a character of C and write $\phi(h_i) = \alpha_i$, $i = 1, \ldots, n$ and $\phi(r) = \rho$. Then as a result of $(\phi \otimes \phi)\Delta = \phi$ we obtain:

$$(\beta_1 \alpha_1 + \cdots + \beta_n \alpha_n)\rho = \rho$$
$$(\beta_1^i \alpha_1 + \cdots + \beta_n^i \alpha_n)\rho = \alpha_i, \quad i = 1, \ldots, n.$$

Thus, if ϕ is a nontrivial character, necessarily, $\rho \neq 0$ and then $a^T = (\alpha_1, \ldots, \alpha_n)$ satisfies conditions $a^T b = 1$ and $a^T A = \frac{1}{\rho} a^T$. □

The connection between ECCPs introduced here and EACPs given in [4] goes through the dual structures. (We refer the reader to [1, Theorem 2.3.14] for a more general result on the duality between algebras and coalgebras.)

Theorem 4.7. *Let C be an ECCP. Then C^* with the inherited algebra structure is an EACP.*

Proof. Let $\mathcal{B} = \{h_1, \ldots, h_n, r\}$ be a natural basis of the ECCP C and let $\mathcal{B}^* = \{h_1^*, \ldots, h_n^*, r^*\}$ be the dual basis of C^* (i.e. $h_i^*(h_j) = \delta_{ij}$, $r^*(r) = 1$ and $h_i^*(r) = r^*(h_i) = 0$ for all $i, j = 1, \ldots, n$). Then it follows from Example 3.1 that the multiplication in C^* for the basis elements of $\mathcal{B}^*$ is $h_i^* h_j^* = 0 = r^* r^*$, $i \neq j$, and

$$r^* h_i^* = h_i^* r^* = \frac{1}{2}\Big(\sum_{j=1}^{n} \beta_i^j h_j^* + \beta_i r^*\Big), \quad i = 1, \ldots, n.$$

Hence C^* is an EACP with natural basis $\mathcal{B}^*$. □

Theorem 4.8. *Let A be an EACP with natural basis $\mathcal{B} = \{h_1, \ldots, h_n, r\}$. Then $C = A^*$ with the inherited coalgebra structure is an ECCP w.r.t. the natural basis $\mathcal{B}^* = \{h_1^*, \ldots, h_n^*, r^*\}$.*

Proof. Write $h_i r = r h_i = \frac{1}{2}\sum_{j=1}^{n}(a_{ij}h_j + b_i r)$ for the multiplication in the EACP A, $i = 1, \ldots, n$, and zero otherwise. Then $C = A^*$ inherits a coalgebra structure (see [15]) such that the comultiplication for the elements of $\mathcal{B}^*$ is:

$$\Delta(h_i^*) = \frac{1}{2}\sum_{j=1}^{n} a_{ij}(h_j^* \otimes r^* + r^* \otimes h_j^*),$$

$$\Delta(r^*) = \frac{1}{2}\sum_{j=1}^{n} b_j(h_j^* \otimes r^* + r^* \otimes h_j^*).$$

□

As considered in [12] for general genetic coalgebras, here we will settle the structure of ECCPs in terms of those subcoalgebras of the chicken population.

Definition 4.9. Let C be an ECCP C (with natural basis $\mathcal{B}$). Then:

(i) A subcoalgebra D of C is an *evolution subcoalgebra of the chicken population* C if D is of the form $D = span_K(h_i, r \mid i \in \Lambda)$, $\emptyset \neq \Lambda \subseteq \{1, \ldots, n\}$.
(ii) The ECCP C is said to be a *simple* ECCP is it has no proper evolution subcoalgebras of the chicken population (assuming also comultiplication is not trivial).

Remark 2. Any evolution subcoalgebra D of a chicken population C is also an ECCP itself, with the same roaster and a smaller (but not empty) group of hens. Note that this comes out from the nonempty assumption on the index set Λ, and makes D to retain the defining property of chicken populations. The existence of only one roaster in chicken populations makes all evolution coalgebras of chicken populations irreducible (as chicken populations) since having just one roaster, it becomes impossible to decompose any chicken population into the direct sum of two such subpopulations (each having its own roaster).

Left, right and two-sided coideals of chicken populations can be defined similarly. Note, however, that as coideals are not necessarily subcoalgebras, their bases (as spanning subsets) do not need to contain a chicken subpopulation. A first approach to the structure of ECCPs was given in [13], where it was settled that any $(n+1)$- dimensional ECCP is the direct sum of an n-dimensional and a 1-dimensional coideals. However these coideals are not coideals of the chicken population. Indeed such decomposition was given in term of evolution coideals (i.e. coideals spanned by subsets of the natural basis) but not of the chicken population.

Proposition 4.10. *Let C be an ECCP with natural basis $\mathcal{B} = \{h_1, \ldots, h_n, r\}$. Then $D = span_K(h_i, r \mid i \in \Lambda)$, $\Lambda \subseteq \{1, \ldots, n\}$ is an evolution subcoalgebra of the chicken population C if and only if D^* is an evolution subalgebra of the EACP C^*. Hence C is simple (as ECCP) if and only if C^* is irreducible (as EACP).*

Proof. It follows from Theorem 4.7 and Theorem 4.8. For the last assertion see [7, Definition 4.5] □

5. The structure of evolution coalgebras of chicken populations.

In this section we introduce the notion of digraph attached to an ECCP and use it to characterize simple ECCPs. In what follows we will assume that C is an ECCP with natural basis $\mathcal{B} = \{h_1, \ldots, h_n, r\}$ and comultiplication as given in Definition 4.1. We refer the reader to [11] for basic notions on digraphs attached to coalgebras with genetic realization (see also [3] for digraphs attached to evolution algebras).

Definition 5.1. We define the *digraph attached to an ECCP C relative to $\mathcal{B}$* to be the (directed) graph $\Gamma(C, \mathcal{B}) = (V, E)$ with set of vertices $V = \{0, 1, \ldots, n\}$ and set of directed edges (or arcs):

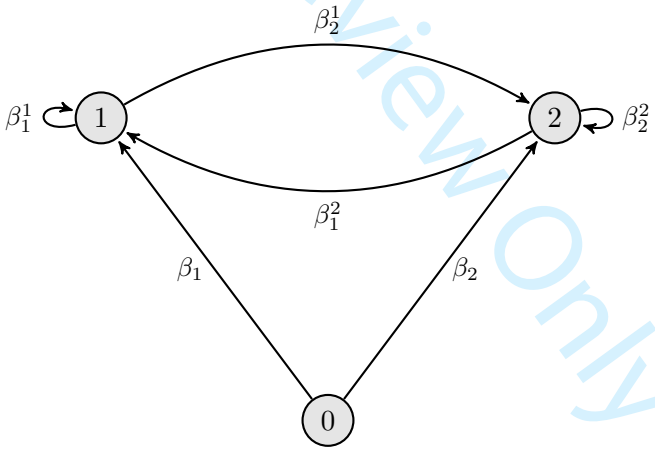
$$E = \Big\{(k, i) \in V \times V \mid \beta_i^k \neq 0,\ i, k \neq 0\Big\} \bigcup \Big\{(0, i) \in V \times V \mid \beta_i \neq 0,\ i \neq 0\Big\}.$$

Remark 3. (i) Vertices $1, \ldots, n$ represent the "hens", while vertex 0 represents the roaster r. Clearly this vertex is a source, with arcs only existing but never entering at it.

(ii) $\Gamma(C, \mathcal{B})$ may contain loops (i.e. arcs of the form uu for some $u \in V$) only for those vertices in $V - \{0\}$.

(iii) $\Gamma(C, \mathcal{B})$ is naturally weighted by the matrix $P = [A \mid b]$ associated to C. Indeed it suffices to consider $\omega(k, i) = \beta_i^k$ and $\omega(0, i) = \beta_i$ for all $i, k = 1, \ldots, n$. Then the map $\omega : E \to K$ defines a weighting on $\Gamma(C, \mathcal{B})$.

Example 5.2. Consider a $(2 + 1)$-dimensional ECCP C with natural basis $\mathcal{B} = \{h_1, h_2, r\}$. Then $\Gamma(C, \mathcal{B})$ is



To be more accurate the digraph associated to any $(2 + 1)$-dimensional ECCP is a subdigraph of the digraph $\Gamma(C, \mathcal{B})$ above, after taking into account that the vanishing of any of the comultiplication structure constants should lead us to discard the corresponding arc.

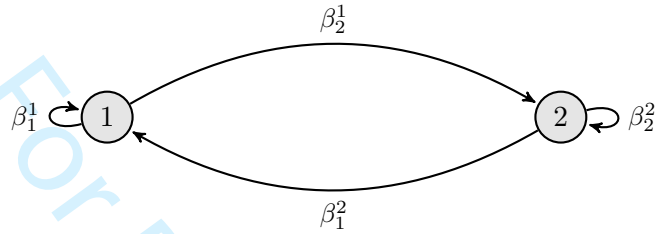
Definition 5.3. We define the h-subdigraph $\Gamma_h(C, \mathcal{B})$ of $\Gamma(C, \mathcal{B})$ to be the subdigraph of $\Gamma(C, \mathcal{B})$ with vertices $V_h = V - \{0\}$ and arcs

$$E_h = \Big\{(k, i) \in V \times V \mid \beta_i^k \neq 0,\ i, k \neq 0\Big\}.$$

Proposition 5.4. *$\Gamma_h(C,\mathcal{B})$ is an induced subdigraph of $\Gamma(C,\mathcal{B})$. Moreover ω induces a weighting in $\Gamma_h(C,\mathcal{B})$.*

Proof. Recall that a subgraph H of a graph G is an induced subgraph if all edges in G linking vertices in H belong to H. The fact that the restriction of ω to $\Gamma_h(C,\mathcal{B})$ induces a weighting on $\Gamma_h(C,\mathcal{B})$ is straightforward. □

Example 5.5. Let $\Gamma(C,\mathcal{B})$ be as in Example 5.2. Then the corresponding h-subdigraph $\Gamma_h(C,\mathcal{B})$ is:



Let us denote $\Gamma = \Gamma(C,\mathcal{B})$ and by Γ_h the corresponding h-subdigraph of Γ. We recall that for any $i \in V$, $N_\Gamma(i) = \{j \in V \mid (i,j) \in E\}$ denotes the set of all *neighbors* of i in Γ. Note that for all $i \in V$, we have $N_\Gamma(i) \subseteq V - \{0\}$. Moreover $N_\Gamma(i) = N_{\Gamma_h}(i)$, for all $i \in V - \{0\}$, as a result of being Γ_h an induced subgraph of Γ (see Proposition 5.4).

Following [11] it is not difficult to establish a correspondence between evolution subcoalgebras of the chicken population C and certain induced subdigraphs of $\Gamma(C,\mathcal{B})$.

Proposition 5.6. *Let C be an ECCP with natural basis $\mathcal{B}$ and let $D = span_K(h_i, r \mid i \in \Lambda)$, $\Lambda \subseteq \{1,\dots,n\}$, be an evolution subcoalgebra of the chicken population. Then $\Gamma_D = (V_D, E_D, \omega_D)$ where:*

(i) $V_D = \{0, i \mid i \in \Lambda\}$,

(ii) $E_D = \Big\{(k,i) \mid \beta_i^k \neq 0,\ i,k \in \Lambda\Big\} \bigcup \Big\{(0,i) \mid \beta_i \neq 0,\ i \in \Lambda\Big\}$,

(iii) $\omega_D : E_D \to K$ *given by* $\omega_D(k,i) = \beta_i^k$ *and* $\omega_D(0,i) = \beta_i$, *for all* $i,k \in \Lambda$,

is a weighted induced subdigraph of $\Gamma(C,\mathcal{B})$ with $\omega_D = \omega_{|D}$. Conversely, for any induced subdigraph Γ' of $\Gamma(C,\mathcal{B})$ with $0 \in V(\Gamma')$ and such that $N_{\Gamma'}(k) = N_{\Gamma(C,\mathcal{B})}(k)$ for all $k \in V(\Gamma')$, the linear span $D = span_K(k \in V(\Gamma'))$ is an evolution subcoalgebra of the chicken population $\mathcal{C}$.

Proof. A proof similar to that of [11, Proposition 6.1, Corollary 6.2] works here to prove that Γ_D is a weighted induced subdigraph of $\Gamma(C,\mathcal{B})$. The second assertion follows as in [11, Theorem 6.3]. □

Example 5.7. Consider the following (1+1)-dimensional ECCPs:

(i) C_1 with natural basis $\mathcal{B} = \{h, r\}$ and comultiplication:

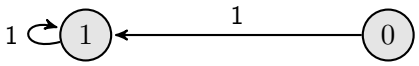
$$\Delta(h) = h \otimes r + r \otimes h,$$
$$\Delta(r) = 0;$$

(ii) C_2 with natural basis $\mathcal{B} = \{h, r\}$ and comultiplication:

$$\Delta(h) = \frac{1}{2}\big(h \otimes r + r \otimes h\big),$$
$$\Delta(r) = \frac{1}{2}\big(h \otimes r + r \otimes h\big).$$

Then the corresponding attached digraphs Γ_i, $i = 1, 2$, are, respectively:

and



We note here that duals of ECCPs C_1 and C_2 above are the EACPs appearing in [4, Proposition 6.1] (see also [14, Propositon 4.1]).

A (directed) walk $u - v$ (of length k) in a digraph is a sequence of vertices $u = u_0u_1 \cdots u_k = v$, with u_{i-1} adjacent to u_i (i.e. $u_{i-1}u_i$ is an arc). A digraph is *strongly connected* if there are $u - v$ and $v - u$ walks for all vertices u, v. Otherwise the digraph is said to be *disconnected*. If only the underlying non-directed graph is connected the digraph is *weakly connected*.

Proposition 5.8. *Digraphs attached to a chicken population are not strongly connected.*

Proof. It suffices to note that, being vertex 0 a source, $\Gamma(C, \mathcal{B})$ contains no walks finishing at vertex 0. □

In what follows we will denote by Γ the digraph $\Gamma(C, \mathcal{B})$ attached to the ECCP C w.r.t the natural basis $\mathcal{B} = \{h_1, \ldots, h_n, r\}$ and by Γ_h its induced h-subdigraph (see Definition 5.3).

Theorem 5.9. *Let C be an ECCP with natural basis $\mathcal{B}$. Assume that $\Delta(x) \neq 0$ for all $x \in \mathcal{B}$. Then, C is simple as ECCP if and only if Γ_h is strongly connected.*

This theorem is based on the following technical lemmas whose proofs work similarly to those of the corresponding results given in [11] for arbitrary genetic coalgebras.

Lemma 5.10. *The following statements hold for all $i, k \in \{1, \ldots, n\}$:*

(i) $i \in N_\Gamma(k) \Rightarrow h_i \in subcoalg(h_k)$.
(ii) $i \in N_\Gamma(0) \Rightarrow h_i \in subcoalg(r)$.

Here $subcoalg(x)$ denotes the smallest (ordinary) subcoalgebra of C containing the element x (see 3.2).

Proof. Similar to [11, Proposition 6.5] and [11, Corollary 6.8]. □

Lemma 5.11. *Let $i - j$, $j \neq 0$, be a directed walk in Γ of length $s \geq 2$ for some $i, j \in \{0, 1, \ldots, n\}$. Then:*

(i) If $i = 0$, then $j \neq i$ and there exist indexes $j_1, j_2, \ldots, j_{s-1} \in \{1, \ldots, n\}$ such that $\beta_{j_1}\beta^{j_1}_{j_2} \ldots \beta^{j_{s-2}}_{j_{s-1}}\beta^{j_{s-1}}_{j} \neq 0$.

(ii) If $i \neq 0$, then there exist indexes $j_1, j_2, \ldots, j_{s-1} \in \{1, \ldots, n\}$ such that $\beta^{i}_{j_1}\beta^{j_1}_{j_2} \ldots \beta^{j_{s-2}}_{j_{s-1}}\beta^{j_{s-1}}_{j} \neq 0$.

Conversely if such a series of indexes exists in $\{1, \ldots, n\}$, then there exists a $i - j$ walk in Γ.

Proof. The proof works as that of [11, Lemma 6.4]. □

Lemma 5.12. *Let $i, j \in \{0, 1, \ldots, n\}$. If there exists a $i - j$ walk in Γ then $j \neq 0$ and $h_j \in subcoalg(h_i)$. In particular, $h_j \in subcoalg(h_i)$ for all $0 \neq j \in N_\Gamma(i)$. (Write here $h_0 = r$.)*

Proof. See the proof of [11, Corollary 6.6] and [11, Corollary 6.8]. □

Proposition 5.13. *Any subcoalgebra of C generated by a natural basis element (i.e. any subcoalgebra of C of the form $subcoalg(x)$, $x \in \mathcal{B}$) with $\Delta(x) \neq 0$ contains an evolution subcoalgebra of the chicken population C.*

Proof. Similar to the proof of [11, Proposition 6.9]. We remark that here assumption $\Delta(x) \neq 0$ plays the role of the genetic realization assumption appearing in [11]. □

Remark 4. Note that any natural basis element $x \in \mathcal{B}$ with trivial comultiplication, i.e. such that $\Delta(x) = 0$, is a source in Γ.

Finally we have all technical requirements to cope with Theorem 5.9, whose proof follows the main guidelines of [11, Theorem 6.10] and [11, Theorem 6.11].

Proof. *Proof of Theorem 5.9.* Assume first that C is simple as ECCP. We claim that there exists a $i - j$ walk in Γ_h for all $i, j \in \{1, \ldots, n\}$.

Let $subcoalg(h_i)$ be the (ordinary) subcoalgebra of C generated by h_i. Then, since $\Delta(h_i) \neq 0$, $subcoalg(h_i)$ contains an evolution subcoalgebra of the chicken population C. But being C simple, as chicken population, we have $C \subseteq subcoalg(h_i)$. Then

$$S = \{p \mid p \in N_\Gamma(k) \quad \text{for some } h_k \in subcoalg(h_i)\} = \{1, \ldots, n\}.$$

Thus there exists a series of indexes $k_0 = i, k_1, \ldots, k_l = j$ $(l \geq 1)$ with $k_m \in N_\Gamma(k_{m-1})$, $m = 1, \ldots, l$, so that k_{m-1} is adjacent to k_m giving rise to a $i - j$ walk in Γ_h. Hence Γ_h is strongly connected.

Conversely, suppose now that Γ_h is strongly connected and let D be a proper evolution subcoalgebra of the chicken population C. Write, reordering the basis if necessary, $D = span_K(h_1, \ldots, h_t, r)$ for some $1 \leq t < n$. Take k_0, such that $t < k_0 \leq n$. Being Γ_h strongly connected, there exists a $t - k_0$ walk in Γ_h implying that $k_0 \in subcoalg(h_t) \subseteq D$, contradicting that $t < k_0$. Hence C is simple as ECCP. □

We conclude by remarking that Theorem 5.9 is not only a simplicity criteria for ECCPs. Indeed as a result of Theorem 4.7, Theorem 4.8 and Proposition 4.10 it also provides us a new tool for studying the structure of EACPs.

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